

Geometrical derivation of the Boltzmann factor

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Abstract

We show that the Boltzmann factor has a geometrical origin. Its derivation follows from the microcanonical picture. The Maxwell-Boltzmann distribution or the wealth distribution in human society are some direct applications of this new interpretation.

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The Maxwellian distribution was derived in Ref. 1 by geometrical arguments over an N -sphere. Following this insight, here we explain the origin of the Boltzmann factor by means of the geometrical properties of an N -hyperplane. This alternative view can be thought as an a priori theoretical argument that explains why the probability of a magnitude, constrained by a conservation law, is given by the exponential distribution when the ensemble or system over which that magnitude is measured is in *thermodynamical equilibrium*.

By thermodynamical equilibrium we mean the macroscopic stationary regime where all the microscopic accessible states of the system are equiprobable. This implies that the deterministic or random time evolution of the system verifies the ergodic hypothesis. This has been proved in very few systems², and hence, in general, it is supposed as an hypothesis in classical statistical mechanics³. If the system do not explore its own phase space with equiprobable results for all the accessible states then the system exhibits some kind of ergodicity breaking and the final distribution will not be the exponential distribution. We can call it a *non-Boltzmann equilibrium*.

Without loss of generality, here we obtain analytically the Boltzmann factor (or Maxwell-Boltzmann distribution) in a model where the ergodic hypothesis has been computationally proved. The method was developed in Ref. 1 and it is grounded on geometrical arguments.

The model was proposed by Dragulescu and Yakovenko⁴ in order to explain the distribution of wealth in human society. Thus it is known that the incomes of 90% of the population in western societies can be fitted by a exponential distribution.⁵ Supposing initially equity in a set of many agents, that is, all of them have the same initial quantity of money, the model establishes random binary interactions in which agents exchange money but conserve its total amount. It is computationally found that the system asymptotically tends toward the exponential distribution. This means that, in this case, the evolution mechanisms proposed by Dragulescu and Yakovenko for the money exchange among agents verify the ergodic hypothesis and, then, its final distribution is the exponential distribution. (Let us say that other non-Boltzmann random mechanisms for the money exchange are also proposed in Ref. 4. These mechanisms imply the breaking of the ergodic hypothesis and therefore the asymptotic evolution of the system toward non-Boltzmann equilibria).

We start by assuming N agents, each one with coordinate (money) x_i , $i = 1, \dots, N$, with

$x_i \geq 0$, and a total amount of money E that is conserved:

$$x_1 + x_2 + \cdots + x_{N-1} + x_N = E. \quad (1)$$

Under the evolution rules proposed in Ref. 4, this isolated system evolves on the positive part of an equilateral N -hyperplane (i.e., the surface formed for all those points with $x_i \geq 0$, for all i). The formula for the surface area $S_N(E)$ of an equilateral N -hyperplane of side E is

$$S_N(E) = \frac{\sqrt{N}}{(N-1)!} E^{N-1}. \quad (2)$$

(See appendix A for the derivation of this formula). If the ergodic hypothesis is assumed, each state of the microcanonical ensemble (that is, each point on the N -hyperplane) is equiprobable, then the probability $f(x_i)dx_i$ of finding the agent i with money x_i is proportional to the surface area formed by all the points on the N -hyperplane having the i th-coordinate equal to x_i . Our objective is to show that $f(x_i)$ is the Boltzmann factor (or the Maxwell-Boltzmann distribution), with the normalization condition

$$\int_0^E f(x_i)dx_i = 1. \quad (3)$$

If the i th agent has coordinate x_i , the $N-1$ remaining agents share the money $E-x_i$ on the $(N-1)$ -hyperplane

$$x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_N = E - x_i, \quad (4)$$

whose surface area is $S_{N-1}(E-x_i)$. If we define the coordinate θ_N (see appendix A) as satisfying

$$\sin \theta_N = \sqrt{\frac{N-1}{N}}, \quad (5)$$

it can be easily proved that

$$S_N(E) = \int_0^E S_{N-1}(E-x_i) \frac{dx_i}{\sin \theta_N}. \quad (6)$$

Hence, the surface area of the N -hyperplane for which the i th coordinate is between x_i and $x_i + dx_i$ is $S_{N-1}(E-x_i)dx_i/\sin \theta_N$. We rewrite the surface area as a function of x_i , normalize it to satisfy Eq. (3), and obtain

$$f(x_i) = \frac{1}{S_N(E)} \frac{S_{N-1}(E-x_i)}{\sin \theta_N}, \quad (7)$$

whose final form, after some calculation is

$$f(x_i) = (N-1)E^{-1} \left(1 - \frac{x_i}{E}\right)^{N-2}, \quad (8)$$

If we call ϵ the mean wealth per agent, $E = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{x_i}{E}\right)^{N-2} \simeq e^{-x_i/\epsilon}. \quad (9)$$

The Boltzmann factor $e^{-x_i/\epsilon}$ is found when $N \gg 1$ but, even for small N , it can be a good approximation for agents with low wealth. After substituting Eq. (9) into Eq. (8), we obtain the Maxwell-Boltzmann distribution in the asymptotic regime $N \rightarrow \infty$ (which also implies $E \rightarrow \infty$):

$$f(x)dx = \frac{1}{\epsilon} e^{-x/\epsilon} dx, \quad (10)$$

where the index i has been removed because the distribution is the same for each agent, and thus the wealth distribution can be obtained by averaging over all the agents.

Depending on the physical situation the mean wealth per agent ϵ takes different expressions and interpretations. For instance, in the case of an isolated one-dimensional gas, the x_i variable represents the energy of the particle i , thus we can calculate the dependence of ϵ on the temperature, which in the microcanonical ensemble is defined by differentiating the entropy with respect to the energy. The entropy can be written as $S = -kN \int_{-\infty}^{\infty} f(x) \ln f(x) dx$, where $f(x)$ is given by Eq. (10) and k is the Boltzmann constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = kN \ln \left(\frac{E}{N}\right) + kN. \quad (11)$$

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{kN}{E} = \frac{k}{\epsilon}. \quad (12)$$

Thus $\epsilon = kT$, and the Maxwell-Boltzmann distribution is obtained as it is usually given in the literature:

$$f(x)dx = \frac{1}{kT} e^{-x/kT} dx. \quad (13)$$

In conclusion, we have shown that the Boltzmann factor describes the general statistical behavior of each small part of a multi-component system whose components or parts are given by a set of random variables that satisfy a conservation law (condition (1)), and that run in an equiprobable manner over all the states of its own phase space.

APPENDIX A: Formula for the surface of an equilateral N -hyperplane

Here we derive the formula for the calculation of the surface $S_N(E)$ of an equilateral N -hyperplane, Π_N , embedded in \mathfrak{R}^N and formed by all the points (x_1, x_2, \dots, x_N) , with $x_i \geq 0$ for all $i = 1, 2, \dots, N$, verifying the equation

$$x_1 + x_2 + \dots + x_{N-1} + x_N = E, \quad (1)$$

with E a real constant.

First we define the angle θ_N as follows. The unitary perpendicular vector to the hyperplane Π_N in \mathfrak{R}^N , ω_\perp , has coordinates $\frac{1}{\sqrt{N}}(1, 1, \dots, 1, 1)$. The unitary vector in the x_N -direction, v_{x_N} , has coordinates $(0, 0, \dots, 0, 1)$. The projection of ω_\perp over v_{x_N} is given by the scalar product of both vectors:

$$\cos \theta_N = \omega_\perp \cdot v_{x_N} = \frac{1}{\sqrt{N}}, \quad (2)$$

that defines the angle θ_N . Let us observe the curious property of this angle:

$$\frac{\cos \theta_N}{\cos \theta_{N-1}} = \sin \theta_N. \quad (3)$$

The surface $S_N(E)$ in \mathfrak{R}^N is the measure of a set of points with $N - 1$ dimensions, just the set of all those points (x_1, x_2, \dots, x_N) , with $x_i \geq 0$, verifying Eq. (1). The one-to-one projection of this set of points on the x_N -direction generates a set of points of $N - 1$ dimensions in \mathfrak{R}^{N-1} , just the set of all those points in \mathfrak{R}^{N-1} satisfying the inequality:

$$x_1 + x_2 + \dots + x_{N-1} \leq E. \quad (4)$$

The volume of this set of points $V_{N-1}(E)$ is the volume of an $(N - 1)$ -dimensional pyramid formed by N vertices linked by $N - 1$ perpendicular sides of length E . It is straightforward to see that the formula for this volume is:

$$V_{N-1}(E) = \frac{E^{N-1}}{(N-1)!}. \quad (5)$$

Moreover, $V_{N-1}(E)$ and $S_N(E)$ can be calculated explicitly when $N = 1, 2, 3, 4$. They verify the relationship:

$$V_{N-1}(E) = S_N(E) \cdot \cos \theta_N. \quad (6)$$

This expresses an exact connection between the surface $S_N(E)$ of the N -hyperplane in \Re^N and its projection in the x_N -direction over the $(N-1)$ -dimensional volume $V_{N-1}(E)$ in \Re^{N-1} for those low dimensional cases. Extrapolating this behavior for all N , we conclude that

$$S_N(E) = \frac{V_{N-1}(E)}{\cos \theta_N} = \frac{\sqrt{N}}{(N-1)!} E^{N-1}. \quad (7)$$

as it has been used in Eq. (2) of the main text.

Although we can not claim the mathematical proof of formula (7), the Boltzmann factor is exactly obtained in the limit $N \gg 1$ after taking this formula as starting point of all the line of reasoning developed in the main text. This result can be used like *a posteriori* argument that proves the correctness of expression (7).

APPENDIX B: A possible generalization: an open problem

In this work, we have shown that an ensemble of positive variables (x_1, x_2, \dots, x_N) verifying

$$x_1 + x_2 + \dots + x_{N-1} + x_N = E, \quad (1)$$

with an adequate mechanism assuring the ergodic hypothesis, i.e., the equiprobability of all the possible states (x_1, x_2, \dots, x_N) on the hypersurface of the phase space, presents the exponential distribution of the generic variable x over the ensemble when $N \rightarrow \infty$,

$$f(x)dx \sim \epsilon^{-1} e^{-x/\epsilon} dx, \quad (2)$$

with $E = N\epsilon$ and ϵ the mean value of the real constant E over the collectivity.

In the work with Ref. 1, it was shown that an ensemble of positive variables (x_1, x_2, \dots, x_N) verifying

$$x_1^2 + x_2^2 + \dots + x_{N-1}^2 + x_N^2 = E, \quad (3)$$

with an adequate mechanism assuring the ergodic hypothesis, i.e., the equiprobability of all the possible states (x_1, x_2, \dots, x_N) on the hypersurface of the phase space, presents the Gaussian distribution of the generic variable x over the ensemble when $N \rightarrow \infty$,

$$f(x)dx \sim \epsilon^{-1/2} e^{-x^2/2\epsilon} dx, \quad (4)$$

with $E = N\epsilon$ and ϵ the mean value of the real constant E over the collectivity.

The general question that we want to bring to the reader is the following. Let b be a real constant. If we have a set of positive variables (x_1, x_2, \dots, x_N) verifying

$$x_1^b + x_2^b + \dots + x_{N-1}^b + x_N^b = E \quad (5)$$

with an adequate mechanism assuring the ergodic hypothesis, i.e., the equiprobability of all the possible states (x_1, x_2, \dots, x_N) on the hypersurface of the phase space, will we have for the generic variable x the distribution

$$f(x)dx \sim \epsilon^{-1/b} e^{-x^b/b\epsilon} dx, \quad (6)$$

when we average over the ensemble in the limit $N \rightarrow \infty$?. The answer to this question, as far as we know, is an open problem.

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